

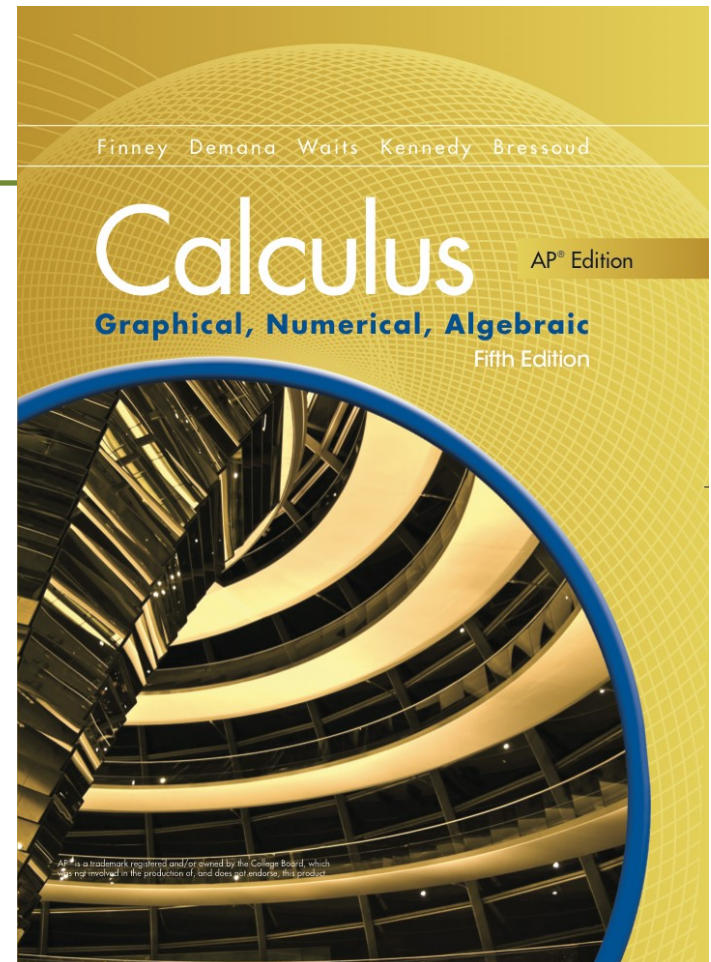
Chapter 2

Limits and Continuity



Section 2.2

Limits Involving Infinity



Quick Review

In Exercises 1 – 4, find ff^1 and graph f , f^{-1} and $y=x$ in the same viewing window.

1. $f(x) = 2x - 3$

2. $f(x) = e^x$

Quick Review

3. $f(x) = \tan^{-1} x$

4. $f(x) = \cot^{-1} x$

Quick Review

In Exercises 5 and 6, find the quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

5. $f(x) = 2x^3 - 3x^2 + x - 1$, $g(x) = 3x^3 + 4x - 5$

6. $f(x) = 2x^5 - x^3 + x - 1$, $g(x) = x^3 - x^2 + 1$

Quick Review

In Exercises 7 – 10, write a formula for (a) $f(-x)$ and (b) $f\left(\frac{1}{x}\right)$.

Simplify where possible.

7. $f(x) = \cos x$

8. $f(x) = e^{-x}$

9. $f(x) = \frac{\ln x}{x}$

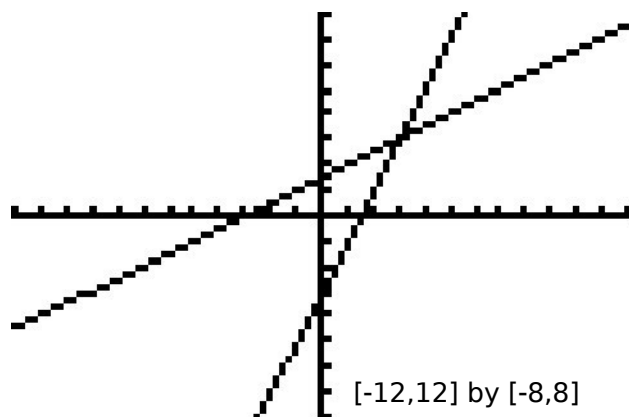
10. $f(x) = \left(x + \frac{1}{x}\right) \sin x$

Quick Review Solutions

In Exercises 1 – 4, find ff^1 and graph f , f^1 and $y=x$ in the same viewing window.

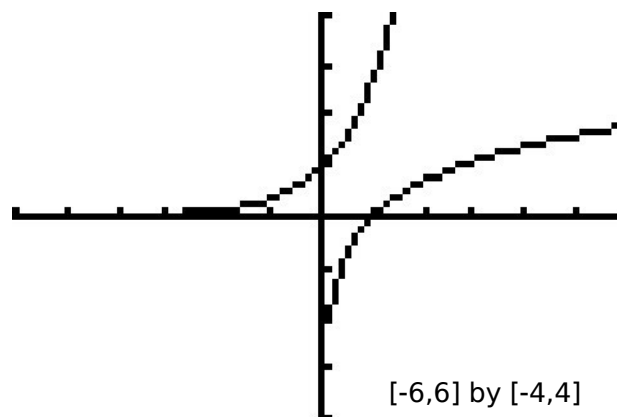
1. $f(x) = 2x - 3$

$$f^1(x) = \frac{x+3}{2}$$



2. $f(x) = e^x$

$$f^1(x) = \ln x$$

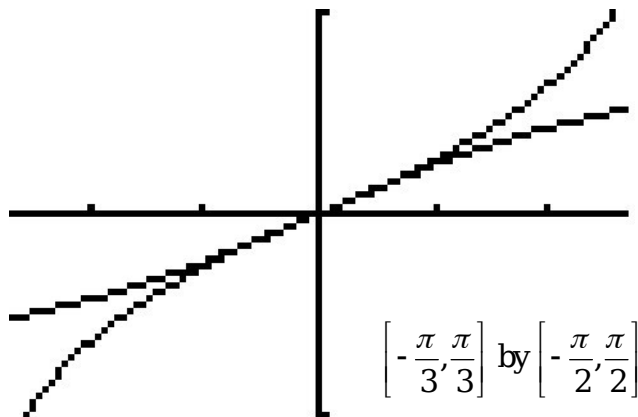


Quick Review Solutions

3. $f(x) = \tan^{-1} x$

$$f^{-1}(x) = \tan x$$

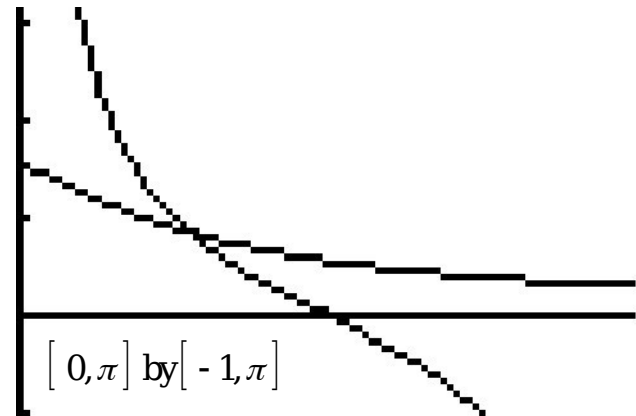
$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$



4. $f(x) = \cot^{-1} x$

$$f^{-1}(x) = \cot x$$

$$0 < x < \pi$$



Quick Review Solutions

In Exercises 5 and 6, find the quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

$$5. \quad f(x) = 2x^3 - 3x^2 + x - 1, \quad g(x) = 3x^3 + 4x - 5$$
$$q(x) = \frac{2}{3}, \quad r(x) = -3x^2 - \frac{5}{3}x + \frac{7}{3}$$

$$6. \quad f(x) = 2x^5 - x^3 + x - 1, \quad g(x) = x^3 - x^2 + 1$$
$$q(x) = 2x^2 + 2x + 1, \quad r(x) = -x^2 - x - 2$$

Quick Review Solutions

In Exercises 7 – 10, write a formula for (a) $f(-x)$ and (b) $f\left(\frac{1}{x}\right)$.

Simplify where possible.

7. $f(x) = \cos x$

8. $f(x) = e^x$

$$f(-x) = \cos x, \quad f\left(\frac{1}{x}\right) = \cos \frac{1}{x} \qquad (-x) = e^x, \quad f\left(\frac{1}{x}\right) = e^{\frac{1}{x}}$$

9. $f(x) = \frac{\ln x}{x}$ $f(-x) = -\frac{\ln(-x)}{x}, \quad f\left(\frac{1}{x}\right) = -x \ln \frac{1}{x}$

10. $f(x) = \left(x + \frac{1}{x}\right) \sin x$ $f(-x) = \left(x + \frac{1}{x}\right) \sin x, \quad f\left(\frac{1}{x}\right) = \left(x + \frac{1}{x}\right) \sin \frac{1}{x}$

What you'll learn about

- The Squeeze Theorem for limits at infinity
- Asymptotic and unbounded behavior of functions
- End behavior of functions

...and why

Limits can be used to describe the behavior of functions for numbers large in absolute value.

Finite limits as $x \rightarrow \pm \infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.

For example, when we say “the limit of f as x approaches infinity” we mean the limit of f as x moves increasingly far to the right on the number line.

When we say “the limit of f as x approaches negative infinity ($-\infty$)” we mean the limit of f as x moves increasingly far to the left on the number line.

Horizontal Asymptote

The line $y=b$ is a **horizontal asymptote** of the graph of a function $y=f(x)$ if either

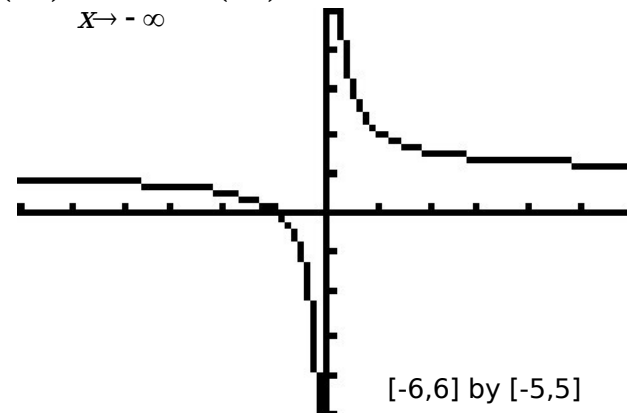
$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b$$

Example Horizontal Asymptote

Use a graph and tables to find (a) $\lim_{x \rightarrow \infty} f(x)$ and (b) $\lim_{x \rightarrow -\infty} f(x)$.

(c) Identify all horizontal asymptotes.

$$f(x) = \frac{x+1}{x}$$



(a) $\lim_{x \rightarrow \infty} f(x) = 1$

(b) $\lim_{x \rightarrow -\infty} f(x) = 1$

(c) Identify all horizontal asymptotes. $y=1$

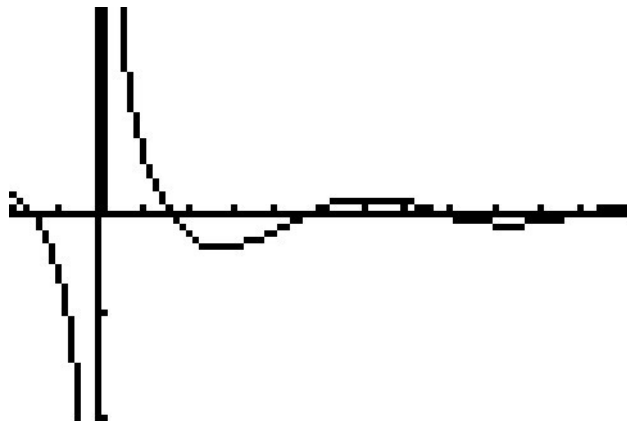
X	Y1
-150	.99333
-100	.99
-50	.98
0	ERROR
50	1.02
100	1.01
150	1.0067

Y1 = (X+1)/X

Example Sandwich Theorem Revisited

The sandwich theorem also holds for limits as $x \rightarrow \pm\infty$.

Find $\lim_{x \rightarrow \infty} \frac{\cos x}{x}$ graphically and using a table of values.



X	Y1
-100	-.0086
100	.00862
300	-7E-5
500	-.0018
700	-.0012
900	7.4E-5
1100	8.2E-4

Y1 = (cos(X))/X

The graph and table suggest that the function oscillates about the x -axis.

Thus $y=0$ is the horizontal asymptote and $\lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0$

Properties of Limits as $x \rightarrow \pm \infty$

If L , M and k are real numbers and

$$\lim_{x \rightarrow \pm \infty} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow \pm \infty} g(x) = M, \text{ then}$$

1. *Sum Rule* $\lim_{x \rightarrow \pm \infty} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule* $\lim_{x \rightarrow \pm \infty} (f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits

Properties of Limits as $x \rightarrow \pm \infty$

3. *Product Rule* $\lim_{x \rightarrow \pm \infty} (f(x) \cdot g(x)) = L \cdot M$

The limit of the product of two functions is the product of their limits.

4. *Constant Multiple Rule* $\lim_{x \rightarrow \pm \infty} (k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule* $\lim_{x \rightarrow \pm \infty} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$

The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

Properties of Limits as $x \rightarrow \pm \infty$

6. *Power Rule* If r and s are integers, $s \neq 0$, then

$$\lim_{x \rightarrow \pm \infty} (f(x))^{\frac{r}{s}} = L^{\frac{r}{s}}$$

provided that $L^{\frac{r}{s}}$ is a real number.

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Infinite Limits as $x \rightarrow a$

If the values of a function $f(x)$ outgrow all positive bounds as x approaches a finite number a , we say that $\lim_{x \rightarrow a} f(x) = \infty$. If the values of f become large and negative, exceeding all negative bounds as x approaches a finite number a , we say that $\lim_{x \rightarrow a} f(x) = -\infty$.

Vertical Asymptote

The line $x=a$ is a **vertical asymptote** of the graph of a function $y=f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty$$

Example Vertical Asymptote

Find the vertical asymptotes of the graph of $f(x)$ and describe the behavior of $f(x)$ to the right and left of each vertical asymptote.

$$f(x) = \frac{8}{4 - x^2}$$

The values of the function approach $-\infty$ to the left of $x = -2$.

The values of the function approach $+\infty$ to the right of $x = -2$.

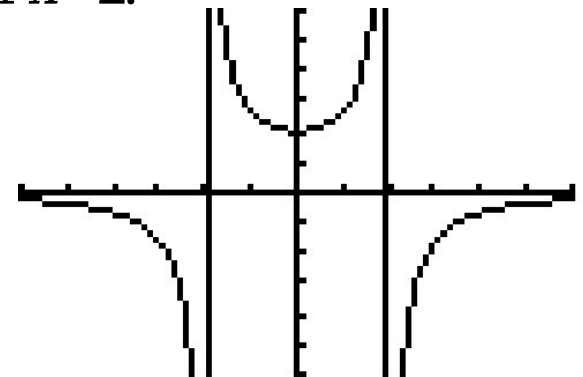
The values of the function approach $+\infty$ to the left of $x = 2$.

The values of the function approach $-\infty$ to the right of $x = 2$.

$$\lim_{x \rightarrow -2^-} \frac{8}{4 - x^2} = -\infty \quad \text{and} \quad \lim_{x \rightarrow -2^+} \frac{8}{4 - x^2} = +\infty$$

$$\lim_{x \rightarrow 2^-} \frac{8}{4 - x^2} = +\infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} \frac{8}{4 - x^2} = -\infty$$

So, the vertical asymptotes are $x = -2$ and $x = 2$



End Behavior Models

The function g is

- (a) a **right end behavior model** for f if and only if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.
- (b) a **left end behavior model** for f if and only if $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = 1$.

End Behavior Models

If one function provides both a left and right end behavior model, it is simply called an **end behavior model**.

In general, $g(x) = a_n x^n$ is an end behavior model for the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0, \quad a_n \neq 0$$

Overall, all polynomials behave like monomials.

Example End Behavior Models

Find an end behavior model for

$$f(x) = \frac{3x^2 - 2x + 5}{4x^2 + 7}$$

Notice that $3x^2$ is an end behavior model for the numerator of f , and $4x^2$ is one for the denominator. This makes

$$\frac{3x^2}{4x^2} = \frac{3}{4} \text{ an end behavior model for } f.$$

End Behavior Models

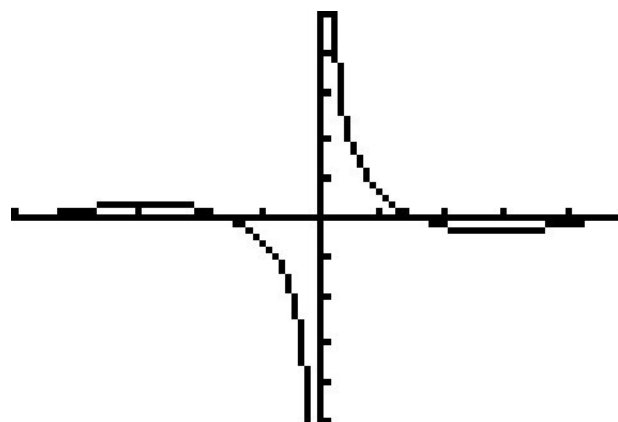
In this example, the end behavior model for f , $y = \frac{3}{4}$ is also a horizontal asymptote of the graph of f . We can use the end behavior model of a rational function to identify any horizontal asymptote. A rational function always has a simple power function as an end behavior model.

Example “Seeing” Limits as $x \rightarrow \pm \infty$

We can investigate the graph of $y = f(x)$ as $x \rightarrow \pm \infty$ by investigating the graph of $y = f\left(\frac{1}{x}\right)$ as $x \rightarrow 0$.

Use the graph of $y = f\left(\frac{1}{x}\right)$ to find $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ for $f(x) = x \cos \frac{1}{x}$.

The graph of $y = f\left(\frac{1}{x}\right) = \frac{\cos x}{x}$ is shown.



$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right) = +\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow 0^-} f\left(\frac{1}{x}\right) = -\infty$$

Quick Quiz Sections 2.1 and 2.2

You may use a graphing calculator to solve the following problems.

1. Find $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}$ if it exists

(A) - 1

(B) 1

(C) 2

(D) 5

(E) does not exist

Quick Quiz Sections 2.1 and 2.2

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(A) - 1

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(C) 2

(D) 5

(E) does not exist

Quick Quiz Sections 2.1 and 2.2

2. Find $\lim_{x \rightarrow 2^+} f(x) = \begin{cases} 3x+1, & x < 2 \\ \frac{5}{x+1}, & x \geq 2 \end{cases}$ if it exists

(A) $\frac{5}{3}$

(B) $\frac{13}{3}$

(C) 7

(D) ∞

(E) does not exist

Quick Quiz Sections 2.1 and 2.2

2. Find $\lim_{x \rightarrow 2^+} f(x) = \begin{cases} 3x+1, & x < 2 \\ \frac{5}{x+1}, & x \geq 2 \end{cases}$ if it exists

(A) $\frac{5}{3}$

(B) $\frac{13}{3}$

(C) 7

(D) ∞

(E) does not exist

Quick Quiz Sections 2.1 and 2.2

3. Which of the following lines is a horizontal asymptote for

$$f(x) = \frac{3x^3 - x^2 + x - 7}{2x^3 + 4x - 5}$$

(A) $y = \frac{3}{2}x$

(B) $y = 0$

(C) $y = \frac{2}{3}$

(D) $y = \frac{7}{5}$

(E) $y = \frac{3}{2}$

Quick Quiz Sections 2.1 and 2.2

3. Which of the following lines is a horizontal asymptote for

$$f(x) = \frac{3x^3 - x^2 + x - 7}{2x^3 + 4x - 5}$$

(A) $y = \frac{3}{2}x$

(B) $y = 0$

(C) $y = \frac{2}{3}$

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(E) $y = \frac{3}{2}$